Dual-Intuitionistic Logic and Some Other Logics

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1 Introduction

This paper is a sequel to Aoyama(2003) and Aoyama(2004). In this paper, we will study various proof-theoretic and model-theoretic properties of the dual-intuitionistic logic DI and some other logics related to it. The logical systems considered in this paper are all in the form of Gentzen's sequent calculus. Models to be studied will be algebraic.

For the classical and intuitionistic logics in the form of Gentzen's sequent calculus, **LK** and **LJ**, we refer the reader to Takeuti(1987).

Definition 1.1 The first-order language for the sequent calculi to be studied in this paper consists of the following symbols:

1. Predicate constants with n argument-places $(n \geq 0)$: $p_0^n, p_1^n, p_2^n, \cdots$
2. Individual constants: $c_0, c_1, c_2, \cdots$
3. Free variables: $a_0, a_1, a_2, \cdots$
4. Bound variables: $x_0, x_1, x_2, \cdots$
5. Logical symbols: $\neg, \wedge, \vee, \rightarrow, \forall, \exists$
6. Auxiliary symbols: (,), ,(comma)

Terms consist of individual constants and free variables. Well-formed formulas (wffs) are defined as usual. When we consider propositional sequent calculi, their language will be defined as follows:

Definition 1.2 The propositional language in this paper will consist of the following symbols:

1. Propositional constants: $p_0, p_1, p_2, \cdots$
2. Logical symbols: $\neg, \wedge, \vee, \rightarrow$
3. Auxiliary symbols: (,)

Well-formed formulas (wffs) of propositional calculi will be defined in the usual way.

2 Dual-Intuitionistic Logic DI

Sequents of **DI** are so restricted that they contain at most one formula in their antecedents.

(1) Axioms : $\varphi \Rightarrow \varphi$

(2) Inference Rules : (Γ consists of at most one formula)

Structural Rules :

$$\text{WL}: \frac{\Rightarrow \Delta}{\varphi \Rightarrow \Delta} \qquad \text{WR}: \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{CR}: \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{ER}: \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$$

$$\text{Cut}: \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda}$$

Logical Rules :

$$\neg\text{L}: \frac{\Rightarrow \Delta, \varphi}{\neg\varphi \Rightarrow \Delta} \qquad \neg\text{R}: \frac{\varphi \Rightarrow \Delta}{\Rightarrow \Delta, \neg\varphi}$$

$$\wedge\text{L}: \frac{\varphi \Rightarrow \Delta}{\varphi \wedge \psi \Rightarrow \Delta} \qquad \wedge\text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\psi \Rightarrow \Delta}{\varphi \wedge \psi \Rightarrow \Delta}$$

$$\vee\text{L}: \frac{\varphi \Rightarrow \Delta \qquad \psi \Rightarrow \Delta}{\varphi \vee \psi \Rightarrow \Delta} \qquad \vee\text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\to\text{L}: \frac{\Rightarrow \Delta, \varphi \qquad \psi \Rightarrow \Lambda}{\varphi \to \psi \Rightarrow \Delta, \Lambda} \qquad \to\text{R}: \frac{\varphi \Rightarrow \Delta, \psi}{\Rightarrow \Delta, \varphi \to \psi}$$

$$\forall\text{L}: \frac{\varphi(t) \Rightarrow \Delta}{\forall x\varphi(x) \Rightarrow \Delta} \qquad \forall\text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\varphi(x)}$$

t is any term (∀L); a is a free variable not occurring in the lower sequent (∀R)

$$\exists\text{L}: \frac{\varphi(a) \Rightarrow \Delta}{\exists x\varphi(x) \Rightarrow \Delta} \qquad \exists\text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\varphi(x)}$$

a is a free variable not occurring in the lower sequent (∃L); t is any term (∃R)

Next, we will consider another logic system which is equivalent to **DI**.

2.1 System DI′

The system equivalent to **DI** is **DI′** which is obtained from **LK** by modifying the four rules of inference $\neg$ L, $\rightarrow$ L, $\rightarrow$ R, and $\exists$ L of **LK**. Thus, there is no restriction on the number of formulas in the antecedent and succedent of a sequent.

(1) Axioms : $\varphi \Rightarrow \varphi$

(2) Inference Rules :

Structural Rules :

$$\text{WL}: \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \text{WR}: \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{CL}: \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \text{CR}: \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{EL}: \frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \qquad \text{ER}: \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$$

$$\text{Cut}: \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical Rules :

$$\neg\ \text{L}: \frac{\Rightarrow \Delta, \varphi}{\neg\varphi \Rightarrow \Delta} \qquad \neg\ \text{R}: \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$\wedge\ \text{L}: \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \wedge\ \text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\vee\ \text{L}: \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \vee\ \text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\rightarrow \text{L}: \frac{\Rightarrow \Delta, \varphi \qquad \psi \Rightarrow \Lambda}{\varphi \rightarrow \psi \Rightarrow \Delta, \Lambda} \qquad \rightarrow \text{R}: \frac{\varphi \Rightarrow \Delta, \psi}{\Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$\forall\ \text{L}: \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \forall\ \text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)}$$

$\forall$ L: t is any term not occurring in Γ

$\forall$ R: a is a free variable not occurring in the lower sequent

$$\exists\ \text{L}: \frac{\varphi(a) \Rightarrow \Delta}{\exists x \varphi(x) \Rightarrow \Delta} \qquad \exists\ \text{R}: \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}$$

$\exists$ L: a is a free variable not occurring in the lower sequent

$\exists$ R: t is any term

The next proposition and remark show some important syntactic properties of **DI** (**DI′**).

Proposition 2.1 *The following sequents are provable in* **DI** (**DI′**):

1. $\Rightarrow \varphi \vee \neg\varphi$
2. $\neg\neg\varphi \Rightarrow \varphi$
3. $\neg\varphi \Rightarrow \neg\neg\neg\varphi$
4. $\varphi \to \psi \Rightarrow \neg\varphi \vee \psi$
5. $\varphi \to \neg\psi \Rightarrow \psi \to \neg\varphi$
6. $\neg\varphi \to \neg\psi \Rightarrow \psi \to \varphi$
7. $\neg(\varphi \wedge \psi) \Rightarrow \neg\varphi \vee \neg\psi$
8. $\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$
9. $\neg(\varphi \vee \psi) \Rightarrow \neg\varphi \wedge \neg\psi$
10. $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
11. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$
12. $\neg\forall x\varphi(x) \Rightarrow \exists x\neg\varphi(x)$
13. $\exists x\neg\varphi(x) \Rightarrow \neg\forall x\varphi(x)$
14. $\neg\forall x\neg\varphi(x) \Rightarrow \exists x\varphi(x)$
15. $\neg\exists x\varphi(x) \Rightarrow \forall x\neg\varphi(x)$

Remark 2.1 The following sequents are not always provable in **DI** (**DI′**):

1. $\varphi \Rightarrow \neg\neg\varphi$
2. $\varphi \wedge \neg\varphi \Rightarrow$
3. $\varphi \Rightarrow \psi \to \varphi$
4. $\varphi \wedge (\varphi \to \psi) \Rightarrow \psi$

We will use the next definition throughout the paper.

Definition 2.1 Suppose $\Gamma = \varphi_1, \varphi_2, \cdots, \varphi_n$. Then, let

$$\bigwedge\Gamma := \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \quad \text{and} \quad \bigvee\Gamma := \varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$$

In the next two propositions, let Γ and Δ be arbitrary finite sequences of formulas. The first is almost trivial:

Proposition 2.2 $\mathbf{DI'} \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{DI'} \vdash \bigwedge\Gamma \Rightarrow \Delta$ iff $\mathbf{DI'} \vdash \bigwedge\Gamma \Rightarrow \bigvee\Delta$

Then we can show the equivalence of the two systems **DI′** and **DI**, the proof of which is elementary:

Proposition 2.3 $\mathbf{DI'} \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{DI} \vdash \bigwedge\Gamma \Rightarrow \Delta$

3 Algebraic Aspects of the Propositional Part of DI′

In this section, we will study algebraic models of the propositional part of **DI′**, which we will write as **DI′p**.

Definition 3.1 A **DI**-algebra $\langle A, \neg, \wedge, \vee, \rightarrow, 0, 1\rangle$ is an algebra satisfying the following properties:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice with a bottom 0 and a top 1.
2. $\neg$ satisfies the condition:
   $a \vee b = 1 \iff \neg b \leq a$
3. $\rightarrow$ satisfies the conditions:
   (1) $a \rightarrow b \leq \neg a \vee b$
   (2) $a \leq b \vee c \implies \neg c \leq a \rightarrow b$

Proposition 3.1 *The operation* $\neg$ *of a* **DI**-*algebra has the following properties*:

1. $\neg 0 = 1, \quad \neg 1 = 0$
2. $a \wedge b = 0 \Longrightarrow b \leq \neg a$
3. $a \vee b = 1 \Longleftrightarrow \neg a \leq b$
4. $a \leq b \Longrightarrow \neg b \leq \neg a$
5. $a \leq b \Longrightarrow a \wedge (b \vee \neg a) = a$
6. $\neg(a \wedge b) = \neg a \vee \neg b, \quad \neg(a \vee b) \leq \neg a \wedge \neg b$
7. $\neg\neg a \leq a, \quad \neg\neg\neg a = \neg a$
8. $\neg a \leq b \Longleftrightarrow \neg b \leq a$

Proposition 3.2 *The operation* $\rightarrow$ *of a* **DI**-*algebra has the following properties*:

1. $a \rightarrow 0 = \neg a$
2. $\neg(a \wedge \neg b) \leq a \rightarrow b$
3. $\neg a \leq a \rightarrow b$
4. $\neg\neg b \leq a \rightarrow b$
5. $a \leq b \Longrightarrow a \rightarrow b = 1$
6. $a \vee (a \rightarrow b) = 1$
7. $a \rightarrow a = 1$
8. $1 \rightarrow a \leq a$
9. $a \vee b = 1 \Longrightarrow a \rightarrow b \leq b$

Proposition 3.3 *The following three conditions are equivalent to each other in a* **DI**-*algebra*:

1. $a \to b \leq \neg a \vee b$
2. $\neg a \vee b \leq c \Longrightarrow a \to b \leq c$
3. $(c \vee a = 1,\ \ b \leq d) \Longrightarrow a \to b \leq c \vee d$

3.1 Algebraic Models of DI′p

Definition 3.2 An algebraic model of **DI′p** is defined to be $\mathcal{M} = \langle M, v\rangle$, where

1. M is a **DI**-algebra $\langle M, \neg, \wedge, \vee, \to, 0, 1\rangle$.
2. v is a function from F to M, where F is the set of all propositional constants of **DI′p**.
3. Interpretation of formulas and sequents of **DI′p** relative to $\mathcal{M} = \langle M, v\rangle$ is defined as follows:

We write the interpretation of a formula φ at v in $\mathcal{M} = \langle M, v\rangle$ as $[\![\varphi]\!]_v^M$.

Formulas :

1. For atomic formulas φ : $[\![\varphi]\!]_v^M := v(\varphi) \in M$
2. For formulas of the form $\neg\psi$: $[\![\neg\psi]\!]_v^M := \neg[\![\psi]\!]_v^M$
3. For formulas of the form $\psi \wedge \chi$: $[\![\psi \wedge \chi]\!]_v^M := [\![\psi]\!]_v^M \wedge [\![\chi]\!]_v^M$
4. For formulas of the form $\psi \vee \chi$: $[\![\psi \vee \chi]\!]_v^M := [\![\psi]\!]_v^M \vee [\![\chi]\!]_v^M$
5. For formulas of the form $\psi \to \chi$: $[\![\psi \to \chi]\!]_v^M := [\![\psi]\!]_v^M \to [\![\chi]\!]_v^M$

Sequents :

1. $[\![\varphi \Rightarrow \psi]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\varphi]\!]_v^M \leq [\![\psi]\!]_v^M \\ 0 & \text{otherwise} \end{cases}$
2. $[\![\quad \Rightarrow \psi]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\psi]\!]_v^M = 1 \\ 0 & \text{otherwise} \end{cases}$
3. $[\![\varphi \Rightarrow \quad]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\varphi]\!]_v^M = 0 \\ 0 & \text{otherwise} \end{cases}$
4. $[\![\varphi_1, \cdots, \varphi_m \Rightarrow \psi_1, \cdots, \psi_n]\!]_v^M := [\![\varphi_1 \wedge \cdots \wedge \varphi_m \Rightarrow \psi_1 \vee \cdots \vee \psi_n]\!]_v^M$

Definition 3.3 Let $\mathcal{M} = \langle M, v\rangle$ be an arbitrary model of **DI′p**.

1. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-*valid*, $\mathcal{M} \models \Gamma \Rightarrow \Delta$ in symbol, if $[\![\Gamma \Rightarrow \Delta]\!]_v^M = 1$ for every v.
2. A sequent $\Gamma \Rightarrow \Delta$ is *valid* , $\models \Gamma \Rightarrow \Delta$ in symbol, if it is $\mathcal{M}$-valid in every model $\mathcal{M}$.

3.2 The Soundness and Completeness of $\mathbf{DI'p}$

In this subsection, we will prove the soundness and completeness theorems of $\mathbf{DI'p}$. The proof of the soundness theorem is routine.

Theorem 3.4 *Let Γ and Δ be arbitrary finite sequences of formulas. Then*

$$\vdash \Gamma \Rightarrow \Delta \implies \models \Gamma \Rightarrow \Delta$$

Before proving the completeness of $\mathbf{DI'p}$, we need to construct the Lindenbaum algebra of $\mathbf{DI'p}$.

Definition 3.4 Let φ and ψ be arbitrary formulas of $\mathbf{DI'p}$.

1. $\varphi \equiv \psi \overset{def}{\iff} (\vdash \varphi \Rightarrow \psi \text{ and } \vdash \psi \Rightarrow \varphi)$
2. $|\varphi| := \{\psi \in \mathcal{F} \mid \varphi \equiv \psi\}$, where $\mathcal{F}$ is the set of all formulas of $\mathbf{DI'p}$
3. $\mathcal{F}/\!\equiv \; := \{|\varphi| \mid \varphi \in \mathcal{F}\}$
3. $|\varphi| \leq |\psi| \overset{def}{\iff} \vdash \varphi \Rightarrow \psi$

Proposition 3.5 *The following hold of the above definition :*

1. *The relation $\leq$ on $\mathcal{F}/\!\equiv$ is well-defined.*
2. *$\langle \mathcal{F}/\!\equiv, \leq \rangle$ is a poset.*
3. *For each $|\varphi|, |\psi| \in \mathcal{F}/\!\equiv$, both $\inf\{|\varphi|, |\psi|\}$ and $\sup\{|\varphi|, |\psi|\}$ exist and the former is written as $|\varphi| \wedge |\psi|$ and the latter as $|\varphi| \vee |\psi|$.*
   Moreover, we have
   $$|\varphi| \wedge |\psi| = |\varphi \wedge \psi|$$
   $$|\varphi| \vee |\psi| = |\varphi \vee \psi|$$
4. *$\langle \mathcal{F}/\!\equiv, \wedge, \vee \rangle$ is a distributive lattice with both a bottom 0 and a top 1 : for any formula φ,*
   $$0 = |\neg(\varphi \to \varphi)| \quad \text{and} \quad 1 = |\varphi \to \varphi|$$
5. *For each $|\varphi| \in \mathcal{F}/\!\equiv$, $\neg|\varphi|$ exists and is equal to $|\neg\varphi|$.*
6. *For each $|\varphi|$ and $|\psi| \in \mathcal{F}/\!\equiv$, $|\varphi| \to |\psi|$ exists and is equal to $|\varphi \to \psi|$.*

By Proposition 3.5, $\langle \mathcal{F}/\!\equiv, \neg, \wedge, \vee, \to, 0, 1 \rangle$ is a $\mathbf{DI}$-algebra, which we will call the "Lindenbaum algebra" of $\mathbf{DI'p}$. Using the Lindenbaum algebra, we can prove the completeness theorem for $\mathbf{DI'p}$:

Theorem 3.6 *Let Γ and Δ be arbitrary finite sequences of formulas. Then*

$$\models \Gamma \Rightarrow \Delta \implies \vdash \Gamma \Rightarrow \Delta$$

Proof Set $\gamma := \bigwedge\!\!\!\bigwedge \Gamma$, $\delta := \bigvee\!\!\!\bigvee \Delta$. Then, assuming $\not\vdash \gamma \Rightarrow \delta$, we show $\not\models \gamma \Rightarrow \delta$. Using the Lindenbaum algebra of $\mathbf{DI'p}$, we define a model $\langle \mathcal{F}/\equiv, v\rangle$ of $\mathbf{DI'p}$ as follows: for each atomic formula φ, set

$$v(\varphi) := |\varphi|$$

Then we can prove, by an easy induction,

Lemma : For each formula φ of $\mathbf{DI'p}$, $[\![\varphi]\!]_v^{\mathcal{F}/\equiv} = |\varphi|$.

Since $\vdash \gamma \Rightarrow \delta \iff |\gamma| \leq |\delta|$, we have $|\gamma| \not\leq |\delta|$. So, by the lemma above,

$$[\![\gamma]\!]_v^{\mathcal{F}/\equiv} \not\leq [\![\delta]\!]_v^{\mathcal{F}/\equiv}, \text{ i.e., } [\![\gamma \Rightarrow \delta]\!]_v^{\mathcal{F}/\equiv} \neq 1$$

Therefore, $\not\models \gamma \Rightarrow \delta$. □

4 Formal Logic System LB

4.1 System LB

Sequent calculus **LB** is obtained from **LK** by restricting each sequent of **LK** so that it contains at most one formula in its antecedent and succedent. Thus **LB** is the following system:

(1) Axioms : $\varphi \Rightarrow \varphi$

(2) Inference Rules : (Γ and Δ contain at most one wff)

Structural Rules

WL : $\dfrac{\Rightarrow \Delta}{\varphi \Rightarrow \Delta}$ WR : $\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}$

Cut : $\dfrac{\Gamma \Rightarrow \varphi \qquad \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

Logical Rules

$\neg$ L : $\dfrac{\Rightarrow \varphi}{\neg\varphi \Rightarrow}$ $\neg$ R : $\dfrac{\varphi \Rightarrow}{\Rightarrow \neg\varphi}$

$\wedge$ L : $\dfrac{\varphi \Rightarrow \Delta}{\varphi \wedge \psi \Rightarrow \Delta}$ $\wedge$ R : $\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$

$\dfrac{\psi \Rightarrow \Delta}{\varphi \wedge \psi \Rightarrow \Delta}$

$\vee$ L : $\dfrac{\varphi \Rightarrow \Delta \qquad \psi \Rightarrow \Delta}{\varphi \vee \psi \Rightarrow \Delta}$ $\vee$ R : $\dfrac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}$

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

$\to$ L : $\dfrac{\Rightarrow \varphi \qquad \psi \Rightarrow \Delta}{\varphi \to \psi \Rightarrow \Delta}$ $\qquad$ $\to$ R : $\dfrac{\varphi \Rightarrow \psi}{\Rightarrow \varphi \to \psi}$

$\forall$ L : $\dfrac{\varphi(t) \Rightarrow \Delta}{\forall x \varphi(x) \Rightarrow \Delta}$ t is any term

$\forall$ R : $\dfrac{\Gamma \Rightarrow \varphi(a)}{\Gamma \Rightarrow \forall x \varphi(x)}$ a is a free variable not occurring in the lower sequent

$\exists$ L : $\dfrac{\varphi(a) \Rightarrow \Delta}{\exists x \varphi(x) \Rightarrow \Delta}$ a is a free variable not occurring in the lower sequent

$\exists$ R : $\dfrac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x \varphi(x)}$ t is any term

Proposition 4.1 *The following sequents are provable in* **LB**:

1. $\varphi \Rightarrow \varphi \wedge (\varphi \vee \psi), \qquad \varphi \wedge (\varphi \vee \psi) \Rightarrow \varphi$
2. $\varphi \Rightarrow \varphi \vee (\varphi \wedge \psi), \qquad \varphi \vee (\varphi \wedge \psi) \Rightarrow \varphi$
3. $\varphi \wedge \psi \Rightarrow \varphi \wedge (\neg\varphi \vee \psi), \qquad \varphi \vee (\neg\varphi \wedge \psi) \Rightarrow \varphi \vee \psi$
4. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$
5. $\varphi \vee (\psi \wedge \chi) \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
6. $(\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \Rightarrow \varphi$
7. $\varphi \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$
8. $\varphi \vee \forall x \psi(x) \Rightarrow \forall x(\varphi \vee \psi(x))$ (x *does not appear in* φ)
9. $\exists x(\varphi \wedge \psi(x)) \Rightarrow \varphi \wedge \exists x \psi(x)$ (x *does not appear in* φ)

Proposition 4.2 *If* $\vdash \; \Rightarrow \varphi$*, then* $\vdash \varphi \to \psi \Rightarrow \neg\varphi \vee \psi$*.*

Proposition 4.3 *The following rules of inference hold in* **LB** *:*

1. $\dfrac{\Rightarrow \varphi}{\Rightarrow \neg\neg\varphi}, \qquad \dfrac{\varphi \Rightarrow}{\neg\neg\varphi \Rightarrow}$
2. $\dfrac{\Rightarrow \varphi \quad \varphi \Rightarrow \psi}{\Rightarrow \psi}, \qquad \dfrac{\Rightarrow \varphi \quad \Rightarrow \varphi \to \psi}{\Rightarrow \psi}$
3. $\dfrac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \neg\neg\psi}$, *where* φ *is a theorem of* **LB***, i.e.,* $\vdash \; \Rightarrow \varphi$
4. $\dfrac{\Rightarrow \varphi \quad \psi \Rightarrow \varphi \to \chi}{\psi \Rightarrow \chi}$

4.2 System LB′

We now present a system of sequent calculus which is named **LB′**. Sequents $\Gamma \Rightarrow \Delta$ in **LB′** has no restriction on the number of formulas in Γ and Δ. It is this system:

(1) Axioms： $\varphi \Rightarrow \varphi$

(2) Inference Rules：

Structural Rules

WL : $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ WR : $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

CL : $\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ CR : $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

EL : $\dfrac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}$ ER : $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$

Cut1 : $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda}$ Cut2 : $\dfrac{\Gamma \Rightarrow \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Lambda}$

Logical Rules

$\neg$ L : $\dfrac{\Rightarrow \varphi}{\neg\varphi \Rightarrow}$ $\neg$ R : $\dfrac{\varphi \Rightarrow}{\Rightarrow \neg\varphi}$

$\wedge$ L : $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$ $\wedge$ R : $\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$

$\dfrac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$

$\vee$ L : $\dfrac{\varphi \Rightarrow \Delta \qquad \psi \Rightarrow \Delta}{\varphi \vee \psi \Rightarrow \Delta}$ $\vee$ R : $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

$\dfrac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

$\rightarrow$ L : $\dfrac{\Rightarrow \varphi \qquad \psi \Rightarrow \Delta}{\varphi \rightarrow \psi \Rightarrow \Delta}$ $\rightarrow$ R : $\dfrac{\varphi \Rightarrow \psi}{\Rightarrow \varphi \rightarrow \psi}$

$\forall$ L : $\dfrac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta}$
t is any term

$\forall$ R : $\dfrac{\Gamma \Rightarrow \varphi(a)}{\Gamma \Rightarrow \forall x \varphi(x)}$
a is a free variable not occurring in the lower sequent

$\exists$ L : $\dfrac{\varphi(a) \Rightarrow \Delta}{\exists x \varphi(x) \Rightarrow \Delta}$
a is a free variable not occurring in the lower sequent

$\exists$ R : $\dfrac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}$
t is any term

This system $\mathbf{LB}'$ is equivalent to $\mathbf{LB}$. To show this, we need a proposition.

Proposition 4.4 *For any given sequent* $\Gamma \Rightarrow \Delta$, *we have the following* :

$$\mathbf{LB}' \vdash \Gamma \Rightarrow \Delta \iff \mathbf{LB}' \vdash \bigwedge \Gamma \Rightarrow \bigvee \Delta$$

Then we can show the equivalence of $\mathbf{LB}$ and $\mathbf{LB}'$:

Theorem 4.5 *For any given sequent* $\Gamma \Rightarrow \Delta$, *we have the following* :

$$\mathbf{LB} \vdash \bigwedge \Gamma \Rightarrow \bigvee \Delta \iff \mathbf{LB}' \vdash \bigwedge \Gamma \Rightarrow \bigvee \Delta$$

4.3 LB and Distributive Laws

It is easy to check that the following distributive laws hold in $\mathbf{LB}$:

1. $\varphi \vee (\psi \wedge \chi) \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
2. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$

For the orther distributive laws, we have the following two propositions.

Proposition 4.6 *Let* Dist_1 *be the sequent of the form* "$(\varphi \vee \psi) \wedge (\varphi \vee \chi) \Rightarrow \varphi \vee (\psi \wedge \chi)$". *Then the following holds* :

$$\mathbf{LB} + \mathrm{Dist}_1 \ \vdash \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

Thus, in $\mathbf{LB} + \mathrm{Dist}_1$, *all of the four distributive laws are provable.*

Proof It is enough to prove in $\mathbf{LB} + \mathrm{Dist}_1$ the following four sequents in order:

1. $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge (\psi \vee \varphi)) \wedge (\psi \vee \chi)$
2. $(\varphi \wedge (\psi \vee \varphi)) \wedge (\psi \vee \chi) \Rightarrow \varphi \wedge (\psi \vee (\varphi \wedge \chi))$
3. $\varphi \wedge (\psi \vee (\varphi \wedge \chi)) \Rightarrow (\varphi \vee (\varphi \wedge \chi)) \wedge (\psi \vee (\varphi \wedge \chi))$
4. $(\varphi \vee (\varphi \wedge \chi)) \wedge (\psi \vee (\varphi \wedge \chi)) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

We here check only 4. The rest are easy to prove. Since "$(\varphi \vee (\varphi \wedge \chi)) \wedge (\psi \vee (\varphi \wedge \chi)) \Rightarrow ((\varphi \wedge \chi) \vee \varphi) \wedge ((\varphi \wedge \chi) \vee \psi)$" is provable in $\mathbf{LB}$, we can get the following proof in $\mathbf{LB}$ +Dist_1, using Cut:

$$\dfrac{\dfrac{(\varphi\vee(\varphi\wedge\chi))\wedge(\psi\vee(\varphi\wedge\chi))\Rightarrow((\varphi\wedge\chi)\vee\varphi)\wedge((\varphi\wedge\chi)\vee\psi) \quad ((\varphi\wedge\chi)\vee\varphi)\wedge((\varphi\wedge\chi)\vee\psi)\Rightarrow(\varphi\wedge\chi)\vee(\varphi\wedge\psi)}{(\varphi\vee(\varphi\wedge\chi))\wedge(\psi\vee(\varphi\wedge\chi)) \Rightarrow (\varphi\wedge\chi)\vee(\varphi\wedge\psi)} \quad (\varphi\wedge\chi)\vee(\varphi\wedge\psi) \Rightarrow (\varphi\wedge\psi)\vee(\varphi\wedge\chi)}{(\varphi\vee(\varphi\wedge\chi))\wedge(\psi\vee(\varphi\wedge\chi)) \Rightarrow (\varphi\wedge\psi)\vee(\varphi\wedge\chi)}$$

□

Proposition 4.7 *Let* Dist_2 *be the sequent of the form* "$\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$". *Then the following holds* :

$$\mathbf{LB} + \mathrm{Dist}_2 \ \vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi) \Rightarrow \varphi \vee (\psi \wedge \chi)$$

Thus, in $\mathbf{LB} + \mathrm{Dist}_2$, *all of the four distributive laws are provable.*

Proof It is enough to prove in **LB** + Dist_2 the following four sequents in order:

1. $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \Rightarrow ((\varphi \vee \psi) \wedge \varphi) \vee ((\varphi \vee \psi) \wedge \chi)$
2. $((\varphi \vee \psi) \wedge \varphi) \vee ((\varphi \vee \psi) \wedge \chi) \Rightarrow \varphi \vee ((\varphi \vee \psi) \wedge \chi)$
3. $\varphi \vee ((\varphi \vee \psi) \wedge \chi) \Rightarrow \varphi \vee ((\varphi \wedge \chi) \vee (\psi \wedge \chi))$
4. $\varphi \vee ((\varphi \wedge \chi) \vee (\psi \wedge \chi)) \Rightarrow \varphi \vee (\psi \wedge \chi)$

We here check only 3. "$\chi \wedge (\varphi \vee \psi) \Rightarrow (\chi \wedge \varphi) \vee (\chi \wedge \psi)$" is a Dist_2 and "$(\chi \wedge \varphi) \vee (\chi \wedge \psi) \Rightarrow (\varphi \wedge \chi) \vee (\psi \wedge \chi)$" is provable in **LB**. Thus, we have "$\chi \wedge (\varphi \vee \psi) \Rightarrow (\varphi \wedge \chi) \vee (\psi \wedge \chi)$" by Cut. Since we also have "$(\varphi \vee \psi) \wedge \chi \Rightarrow \chi \wedge (\varphi \vee \psi)$" in **LB**, we can get the following proof in **LB** + Dist_2 :

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee ((\varphi \wedge \chi) \vee (\psi \wedge \chi))} \qquad \dfrac{\dfrac{(\varphi \vee \psi) \wedge \chi \Rightarrow \chi \wedge (\varphi \vee \psi) \qquad \chi \wedge (\varphi \vee \psi) \Rightarrow (\varphi \wedge \chi) \vee (\psi \wedge \chi)}{(\varphi \vee \psi) \wedge \chi \Rightarrow (\varphi \wedge \chi) \vee (\psi \wedge \chi)}}{(\varphi \vee \psi) \wedge \chi \Rightarrow \varphi \vee ((\varphi \wedge \chi) \vee (\psi \wedge \chi))}}{\varphi \vee ((\varphi \vee \psi) \wedge \chi) \Rightarrow \varphi \vee ((\varphi \wedge \chi) \vee (\psi \wedge \chi))}$$

□

4.4 The Relation between LB and DI

We now study an important sysntactic relation between **LB** and **DI**.

Definition 4.1 We define $\mathbf{LB}_1$ to be a system obtained from **LB** by adding the following sequents as additional axioms:

1. $\Rightarrow \varphi \vee \neg\varphi$
2. $\varphi \to \psi \Rightarrow \neg\varphi \vee \psi$
3. $\neg\varphi \Rightarrow \psi \vee \neg(\psi \vee \varphi)$
4. $\neg\varphi \vee \neg\neg\psi \Rightarrow \varphi \to \psi$
5. $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \Rightarrow \varphi \vee (\psi \wedge \chi)$
6. $\forall x(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x\psi(x)$, where x does not appear in φ

Remark 4.1 *We have* $\mathbf{LB} \vdash \neg\varphi \Rightarrow \neg\varphi \vee \neg\neg\psi$ *and* $\mathbf{LB} \vdash \neg\neg\psi \Rightarrow \neg\varphi \vee \neg\neg\psi$. *Thus, from 4 in the above definition, we have*

$$\mathbf{LB} + 4 \vdash \neg\varphi \Rightarrow \varphi \to \psi \quad \textit{and} \quad \mathbf{LB} + 4 \vdash \neg\neg\psi \Rightarrow \varphi \to \psi$$

Proposition 4.8 *The additional axioms 1-6 in the above definition are all provable in* **DI**.

The next is an easy proposition :

Proposition 4.9 *Let* Γ *be a sequence of at most one formula and* Δ *a finite sequence of formulas. Then*

$$\mathbf{DI} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \mathbf{DI} \vdash \Gamma \Rightarrow \bigvee \Delta$$

The next theorem shows the equivalence of **DI** and $\mathbf{LB}_1$:

Theorem 4.10 $\mathbf{DI} = \mathbf{LB}_1$, *i.e.,* $\mathbf{DI} \vdash \Gamma \Rightarrow \bigvee \Delta$ *iff* $\mathbf{LB}_1 \vdash \Gamma \Rightarrow \bigvee \Delta$, *where* Γ *consists of at most one formula.*

Proof The direction $\Longleftarrow$ is clear from Proposition 4.8 and the fact that **LB** is a subsystem of **DI**. For the other direction, we need to show that all of the inference rules of **DI** hold in $\mathbf{LB}_1$. We here check the inference rules $\neg$ L, $\neg$ R, $\rightarrow$ L, and $\rightarrow$ R of **DI**.

(1) $\neg\,\mathrm{L}: \quad \dfrac{\Rightarrow \Delta, \varphi}{\neg\varphi \Rightarrow \Delta}$

For this, it is enough to check the rule

$$\dfrac{\Rightarrow \delta \vee \varphi}{\neg\varphi \Rightarrow \delta},$$

where $\delta := \bigvee \Delta$. This is obtained in $\mathbf{LB}_1$, using the additional axiom 3 of Definition 4.1, as follows:

$$\dfrac{\neg\varphi \Rightarrow \delta \vee \neg(\delta \vee \varphi) \qquad \dfrac{\delta \Rightarrow \delta \qquad \dfrac{\dfrac{\Rightarrow \delta \vee \varphi}{\neg(\delta \vee \varphi) \Rightarrow}}{\neg(\delta \vee \varphi) \Rightarrow \delta}}{\delta \vee \neg(\delta \vee \varphi) \Rightarrow \delta}}{\neg\varphi \Rightarrow \delta}$$

(2) $\neg\,\mathrm{R}: \quad \dfrac{\varphi \Rightarrow \Delta}{\Rightarrow \Delta, \neg\varphi}$

For this, it is enough to check the rule

$$\dfrac{\varphi \Rightarrow \delta}{\Rightarrow \delta \vee \neg\varphi},$$

where $\delta := \bigvee \Delta$. This is obtained in $\mathbf{LB}_1$, using the additional axiom 1 of Definition 4.1, as follows:

$$\dfrac{\Rightarrow \varphi \vee \neg\varphi \qquad \dfrac{\dfrac{\varphi \Rightarrow \delta}{\varphi \Rightarrow \delta \vee \neg\varphi} \qquad \dfrac{\neg\varphi \Rightarrow \neg\varphi}{\neg\varphi \Rightarrow \delta \vee \neg\varphi}}{\varphi \vee \neg\varphi \Rightarrow \delta \vee \neg\varphi}}{\Rightarrow \delta \vee \neg\varphi}$$

(3) $\rightarrow \mathrm{L}: \quad \dfrac{\Rightarrow \Delta, \varphi \quad \psi \Rightarrow \Lambda}{\varphi \rightarrow \psi \Rightarrow \Delta, \Lambda}$

For this, it is enough to check the rule

$$\dfrac{\Rightarrow \delta \vee \varphi \quad \psi \Rightarrow \lambda}{\varphi \rightarrow \psi \Rightarrow \delta \vee \lambda},$$

where $\delta := \bigvee \Delta$ and $\lambda := \bigvee \Lambda$. This is obtained in $\mathbf{LB}_1$, using (1) above and the additional axiom 2 of Definition 4.1, as follows:

$$\dfrac{\varphi \to \psi \Rightarrow \neg\varphi \vee \psi \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \delta \vee \varphi}{\neg\varphi \Rightarrow \delta}}{\neg\varphi \Rightarrow \delta \vee \lambda} \qquad \dfrac{\psi \Rightarrow \lambda}{\psi \Rightarrow \delta \vee \lambda}}{\neg\varphi \vee \psi \Rightarrow \delta \vee \lambda}}{\varphi \to \psi \Rightarrow \delta \vee \lambda}$$

(4) $\to$ R : $\dfrac{\varphi \Rightarrow \Delta, \psi}{\Rightarrow \Delta, \varphi \to \psi}$

For this, it is enough to check the rule

$$\dfrac{\varphi \Rightarrow \delta \vee \psi}{\Rightarrow \delta \vee (\varphi \to \psi)},$$

where $\delta := \bigvee \Delta$. This is obtained in $\mathbf{LB}_1$ from Remark 4.1, (1) and (2) above and also the fact that $\mathbf{LB} \vdash \varphi \vee \psi \vee \chi \Rightarrow \varphi \vee \chi \vee \psi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \delta \vee \psi}{\Rightarrow \delta \vee \psi \vee \neg\varphi} \qquad \dfrac{\dfrac{\delta \vee \psi \Rightarrow \delta \vee \psi}{\delta \vee \psi \Rightarrow \delta \vee \psi \vee (\varphi \to \psi)} \qquad \dfrac{\neg\varphi \Rightarrow \varphi \to \psi}{\neg\varphi \Rightarrow \delta \vee \psi \vee (\varphi \to \psi)}}{\delta \vee \psi \vee \neg\varphi \Rightarrow \delta \vee \psi \vee (\varphi \to \psi)}}{\Rightarrow \delta \vee \psi \vee (\varphi \to \psi)} \qquad \delta \vee \psi \vee (\varphi \to \psi) \Rightarrow \delta \vee (\varphi \to \psi) \vee \psi}{\Rightarrow \delta \vee (\varphi \to \psi) \vee \psi}}{\neg\psi \Rightarrow \delta \vee (\varphi \to \psi)}}{\Rightarrow \delta \vee (\varphi \to \psi) \vee \neg\neg\psi} \qquad \dfrac{\delta \vee (\varphi \to \psi) \Rightarrow \delta \vee (\varphi \to \psi) \qquad \dfrac{\neg\neg\psi \Rightarrow \varphi \to \psi}{\neg\neg\psi \Rightarrow \delta \vee (\varphi \to \psi)}}{\delta \vee (\varphi \to \psi) \vee \neg\neg\psi \Rightarrow \delta \vee (\varphi \to \psi)}}{\Rightarrow \delta \vee (\varphi \to \psi)}$$

□

4.5 The Relation between LB and LJ

We now study an important syntactic relation between **LB** and **LJ**.

Definition 4.2 We define $\mathbf{LB}_2$ to be a system obtained from **LB** by adding the following sequents as additional axioms:

1. $\varphi \wedge \neg\varphi \Rightarrow$
2. $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
3. $\varphi \wedge (\varphi \to \psi) \Rightarrow \psi$
4. $\varphi \wedge \neg(\psi \wedge \varphi) \Rightarrow \neg\psi$
5. $\varphi \wedge ((\psi \wedge \varphi) \to \chi) \Rightarrow \psi \to \chi$
6. $\exists x \varphi(x) \wedge \psi \Rightarrow \exists x(\varphi(x) \wedge \psi)$, where x does not appear in ψ

Proposition 4.11 *The additional axioms 1-6 in the above definition are all provable in* **LJ**.

The next is a well-known proposition:

Proposition 4.12 *Let* Γ *be a finite sequence of formulas and* Δ *a sequence of at most one formula. Then*

$$\mathbf{LJ} \vdash \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \mathbf{LJ} \vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$$

The next theorem shows the equivalence of **LJ** and $\mathbf{LB}_2$:

Theorem 4.13 $\mathbf{LJ} = \mathbf{LB}_2$*, i.e.,* $\mathbf{LJ} \vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$ *iff* $\mathbf{LB}_2 \vdash \bigwedge\!\!\!\bigwedge \Gamma \Rightarrow \Delta$*, where* Δ *consists of at most one formula.*

Proof The direction $\Longleftarrow$ is clear from Proposition 4.11 and the fact that **LB** is a subsystem of **LJ**. For the other direction, we need to show that all of the inference rules of **LJ** hold in $\mathbf{LB}_2$. We here check the cases of $\neg$L, $\neg$R, $\rightarrow$ L, and $\rightarrow$ R of **LJ**.

(1) $\neg\mathrm{L} : \dfrac{\Gamma \Rightarrow \varphi}{\neg\varphi, \Gamma \Rightarrow}$

For this, it is enough to show that the following inference rule holds in $\mathbf{LB}_2$, where $\gamma := \bigwedge\!\!\!\bigwedge \Gamma$:

$$\dfrac{\gamma \Rightarrow \varphi}{\neg\varphi \wedge \gamma \Rightarrow}$$

Using the additional axiom 1 of $\mathbf{LB}_2$, the proof goes like this:

$$\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \varphi}{\neg\varphi \wedge \gamma \Rightarrow \varphi} \qquad \dfrac{\neg\varphi \Rightarrow \neg\varphi}{\neg\varphi \wedge \gamma \Rightarrow \neg\varphi}}{\neg\varphi \wedge \gamma \Rightarrow \varphi \wedge \neg\varphi} \qquad \varphi \wedge \neg\varphi \Rightarrow}{\neg\varphi \wedge \gamma \Rightarrow}$$

(2) $\neg\mathrm{R} : \dfrac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$

For this, it is enough to show that the following inference rule holds in $\mathbf{LB}_2$, where $\gamma := \bigwedge\!\!\!\bigwedge \Gamma$:

$$\dfrac{\varphi \wedge \gamma \Rightarrow}{\gamma \Rightarrow \neg\varphi}$$

Using the additional axiom 4 of $\mathbf{LB}_2$, the proof goes like this:

$$\dfrac{\dfrac{\gamma \Rightarrow \gamma \qquad \dfrac{\dfrac{\varphi \wedge \gamma \Rightarrow}{\Rightarrow \neg(\varphi \wedge \gamma)}}{\gamma \Rightarrow \neg(\varphi \wedge \gamma)}}{\gamma \Rightarrow \gamma \wedge \neg(\varphi \wedge \gamma)} \qquad \gamma \wedge \neg(\varphi \wedge \gamma) \Rightarrow \neg\varphi}{\gamma \Rightarrow \neg\varphi}$$

(3) $\to$ L : $\dfrac{\Gamma \Rightarrow \varphi \qquad \psi, \Pi \Rightarrow \Delta}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta}$, where Δ consists of at most one formula.

For this, it is enough to show that the following inference rule holds in $\mathbf{LB}_2$, where $\gamma := \bigwedge\!\!\!\bigwedge \Gamma$ and $\pi := \bigwedge\!\!\!\bigwedge \Pi$:

$$\frac{\gamma \Rightarrow \varphi \qquad \psi \wedge \pi \Rightarrow \Delta}{(\varphi \to \psi) \wedge \gamma \wedge \pi \Rightarrow \Delta}$$

Using the additional axiom 3 of $\mathbf{LB}_2$, the proof goes like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \varphi}{(\varphi \to \psi) \wedge \gamma \Rightarrow \varphi} \qquad \dfrac{\varphi \to \psi \Rightarrow \varphi \to \psi}{(\varphi \to \psi) \wedge \gamma \Rightarrow \varphi \to \psi}}{(\varphi \to \psi) \wedge \gamma \Rightarrow \varphi \wedge (\varphi \to \psi)} \qquad \varphi \wedge (\varphi \to \psi) \Rightarrow \psi}{(\varphi \to \psi) \wedge \gamma \Rightarrow \psi}}{(\varphi \to \psi) \wedge \gamma \wedge \pi \Rightarrow \psi} \qquad \dfrac{\dfrac{}{\pi \Rightarrow \pi}}{(\varphi \to \psi) \wedge \gamma \wedge \pi \Rightarrow \pi}}{(\varphi \to \psi) \wedge \gamma \wedge \pi \Rightarrow \psi \wedge \pi} \qquad \psi \wedge \pi \Rightarrow \Delta}{(\varphi \to \psi) \wedge \gamma \wedge \pi \Rightarrow \Delta}$$

(4) $\to$ R : $\dfrac{\varphi\,, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$

For this, it is enough to show that the following inference rule holds in $\mathbf{LB}_2$, where $\gamma := \bigwedge\!\!\!\bigwedge \Gamma$:

$$\frac{\varphi \wedge \gamma \Rightarrow \psi}{\gamma \Rightarrow \varphi \to \psi}$$

Using the additional axiom 5 of $\mathbf{LB}_2$, the proof goes like this:

$$\dfrac{\dfrac{\gamma \to \gamma \qquad \dfrac{\dfrac{\varphi \wedge \gamma \Rightarrow \psi}{\Rightarrow (\varphi \wedge \gamma) \to \psi}}{\gamma \Rightarrow (\varphi \wedge \gamma) \to \psi}}{\gamma \Rightarrow \gamma \wedge ((\varphi \wedge \gamma) \to \psi)} \qquad \gamma \wedge ((\varphi \wedge \gamma) \to \psi) \Rightarrow \varphi \to \psi}{\gamma \Rightarrow \varphi \to \psi}$$

□

4.6 Algebraic models of LB′p

We now consider algebraic models of the propositional part of **LB′**, which will be denoted by **LB′p**.

Definition 4.3 An algebra $A = \langle A, \neg, \wedge, \vee, \to, 0, 1 \rangle$ is called an **LB′p**-algebra when the following conditions are satisfied:

1. $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with a bottom element 0 and a top element 1.
2. $\neg$ is a unary operation satisfying the two conditions: for each $a \in A$,

(1) $\neg a = 1 \ \ if \ \ a = 0$
(2) $\neg a = 0 \ \ if \ \ a = 1$

3. $\rightarrow$ is a binary operation satisfying the two conditions: for each $a, b, c \in A$,

   (1) $a \rightarrow b = 1 \ \ if \ \ a \leq b$
   (2) $a \rightarrow b \leq c \ \ if \ \ a = 1 \ and \ b \leq c$

Proposition 4.14 *In an* **LB′p**-*algebra* $A = \langle A, \neg, \wedge, \vee, \rightarrow, 0, 1\rangle$, *the following hold : for each* $a, b, c \in A$,

1. $1 \rightarrow a \leq a$
2. $a \rightarrow 1 = 1$
3. $0 \rightarrow a = 1$
4. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$
5. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
6. $a \wedge b \leq a \wedge (\neg a \vee b)$
7. $a \vee (\neg a \wedge b) \leq a \vee b$

4.6.1 The Lindenbaum algebra of LB′p

We here consider the Lindenbaum algebra of **LB′p**.

Definition 4.4 Using **LB′p**, we define an algebraic system as follows, where F_L is the set of all the wffs of **LB′p** and $\varphi, \psi \in F_L$:

1. $\varphi \equiv \psi \overset{def}{\Longleftrightarrow} (\vdash \varphi \Rightarrow \psi \ \text{ and } \vdash \psi \Rightarrow \varphi)$
2. $|\varphi| := \{\psi \in F_L \mid \varphi \equiv \psi\}$
3. $F_L/_\equiv := \{|\varphi| \mid \varphi \in F_L\}$
4. $|\varphi| \leq |\psi| \overset{def}{\Longleftrightarrow} \vdash \varphi \Rightarrow \psi$

Proposition 4.15 *The algebra* $\langle F_L/_\equiv, \leq\rangle$ *defined by the previous definition has the following properties :*

1. *The relation* $\equiv$ *is an equivalence relation and the relation* $\leq$ *on* $F_L/_\equiv$ *is well-defined.*
2. $\langle F_L/_\equiv, \leq\rangle$ *is a poset with a bottom element* 0 *and a top element* 1.
3. *For each* $|\varphi|, |\psi| \in F_L/_\equiv$, *there exist* $\sup\{|\varphi|, |\psi|\}$ *and* $\inf\{|\varphi|, |\psi|\}$. *Set*

$$|\varphi| \wedge |\psi| := \inf\{|\varphi|, |\psi|\} \quad \text{and} \quad |\varphi| \vee |\psi| := \sup\{|\varphi|, |\psi|\}$$

   Then we have

$$|\varphi| \wedge |\psi| = |\varphi \wedge \psi| \quad \text{and} \quad |\varphi| \vee |\psi| = |\varphi \vee \psi|$$

Thus, $\langle F_L/_\equiv, \wedge, \vee\rangle$ *is a lattice.*

4. *The lattice* $\langle F_L/_\equiv, \wedge, \vee\rangle$ *has a bottom* 0 *and a top* 1 *.*

5. *For any* $\varphi \in F_L$, (1) $|\neg\varphi| = 1$, *if* $|\varphi| = 0$

   (2) $|\neg\varphi| = 0$, *if* $|\varphi| = 1$

6. *For any* $\varphi, \psi, \chi \in F_L$, (1) $|\varphi \to \psi| = 1$, *if* $|\varphi| \leq |\psi|$

   (2) $|\varphi \to \psi| \leq |\chi|$, *if* $|\varphi| = 1$ *and* $|\psi| \leq |\chi|$

Thus, by defining $\neg$ *and* $\to$ *on* $F_L/_\equiv$ *by*

$$\neg|\varphi| := |\neg\varphi| \quad \textit{and} \quad |\varphi| \to |\psi| := |\varphi \to \psi|,$$

we have an **LB′p**-*algebra* $\langle F_L/_\equiv, \neg, \wedge, \vee, \to, 0, 1\rangle$.

4.6.2 Algebraic Models of LB′p

We are now ready to define algebraic models of **LB′p**.

Definition 4.5 $\mathcal{M} = \langle M, v\rangle$ is called an **LB′p**-model if it satisfies the following conditions:

1. M is an **LB′p**-algebra $\langle M, \neg, \wedge, \vee, \to, 0, 1\rangle$.
2. v is a valuation mapping: $v = \mathcal{V} \longrightarrow M$, where $\mathcal{V}$ is the set of all atomic wffs of **LB′p**.
3. For each wff φ of **LB′p**, $[\![\varphi]\!]_v^M$ indicates the truth value of φ in $\langle M, v\rangle$. The truth value of each wff φ of **LB′p** is defined inductively as follows:

   (1) $[\![p]\!]_v^M := v(p)$, where p is an arbitrary atomic wff

   (2) $[\![\neg\varphi]\!]_v^M := \neg[\![\varphi]\!]_v^M$

   (3) $[\![\varphi \wedge \psi]\!]_v^M := [\![\varphi]\!]_v^M \wedge [\![\psi]\!]_v^M$

   (4) $[\![\varphi \vee \psi]\!]_v^M := [\![\varphi]\!]_v^M \vee [\![\psi]\!]_v^M$

   (5) $[\![\varphi \to \psi]\!]_v^M := [\![\varphi]\!]_v^M \to [\![\psi]\!]_v^M$

4. A wff φ is said to be true in $\mathcal{M} = \langle M, v\rangle$, if $[\![\varphi]\!]_v^M = 1$
5. A wff φ is said to be $\mathcal{M}$-valid in an **LB′p**-model $\mathcal{M}$, if $[\![\varphi]\!]_v^M = 1$ for each v of $\mathcal{M} = \langle M, v\rangle$.
6. A wff φ is said to be valid, if φ is $\mathcal{M}$-valid in every **LB′p**-model $\mathcal{M}$.
7. Sequents of **LB′p** are algebraically interpreted as follows:

(1) $[\![\varphi \Rightarrow \psi]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\varphi]\!]_v^M \leq [\![\psi]\!]_v^M \\ 0 & \text{otherwise} \end{cases}$

(2) $[\![\ \Rightarrow \psi]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\psi]\!]_v^M = 1 \\ 0 & \text{otherwise} \end{cases}$

(3) $[\![\varphi \Rightarrow\ ]\!]_v^M := \begin{cases} 1 & \text{if} \quad [\![\varphi]\!]_v^M = 0 \\ 0 & \text{otherwise} \end{cases}$

(4) $[\![\varphi_1, \cdots, \varphi_m \Rightarrow \psi_1, \cdots, \psi_n]\!]_v^M := [\![\varphi_1 \wedge \cdots \wedge \varphi_m \Rightarrow \psi_1 \vee \cdots \vee \psi_n]\!]_v^M$

(5) The truth, $\mathcal{M}$-validity and validity of a sequent are defined as in the case of a wff. When sequent $\Gamma \Rightarrow \Delta$ is valid, we write as $\models \Gamma \Rightarrow \Delta$.

We now have the soundness theorem of $\mathbf{LB'p}$ with respect to $\mathbf{LB'p}$-models, the proof of which is routine:

Theorem 4.16 *For each sequent* $\Gamma \Rightarrow \Delta$ *of* $\mathbf{LB'p}$*, we have*

$$\vdash \Gamma \Rightarrow \Delta \implies \models \Gamma \Rightarrow \Delta$$

We also have the completeness theorem of $\mathbf{LB'p}$ with respect to $\mathbf{LB'p}$-models, the proof of which is similar to the case of $\mathbf{DI'p}$:

Theorem 4.17 *For each sequent* $\Gamma \Rightarrow \Delta$ *of* $\mathbf{LB'p}$*, we have*

$$\models \Gamma \Rightarrow \Delta \implies \vdash \Gamma \Rightarrow \Delta$$

5 Lattice Logic LL

We now consider a formal system of sequent calculus representing complete lattice. We will call the system **LL**.

Each sequent of **LL** is so restricted that it contains exactly one formula in both the antecedent and the succedent. The language of **LL** contains propositional constants $\top$ (Truth or Verum) and $\bot$ (Falsity or Falsum) but it does not contain the logical symbol $\neg$ or $\rightarrow$.

5.1 System LL

(1) Axioms : $\varphi \Rightarrow \varphi, \qquad \varphi \Rightarrow \top, \qquad \bot \Rightarrow \varphi$

(2) Inference Rules :

Structural Rule :

Cut : $\dfrac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \chi}{\varphi \Rightarrow \chi}$

Logical Rules :

$$\wedge\text{L}: \frac{\varphi \Rightarrow \chi}{\varphi \wedge \psi \Rightarrow \chi} \qquad \wedge\text{R}: \frac{\varphi \Rightarrow \psi \quad \varphi \Rightarrow \chi}{\varphi \Rightarrow \psi \wedge \chi}$$

$$\frac{\psi \Rightarrow \chi}{\varphi \wedge \psi \Rightarrow \chi}$$

$$\vee\text{L}: \frac{\varphi \Rightarrow \chi \quad \psi \Rightarrow \chi}{\varphi \vee \psi \Rightarrow \chi} \qquad \vee\text{R}: \frac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \psi \vee \chi}$$

$$\frac{\varphi \Rightarrow \chi}{\varphi \Rightarrow \psi \vee \chi}$$

$$\forall\text{L}: \frac{\varphi(t) \Rightarrow \psi}{\forall x\varphi(x) \Rightarrow \psi} \qquad \forall\text{R}: \frac{\varphi \Rightarrow \psi(a)}{\varphi \Rightarrow \forall x\psi(x)}$$

t is any term; a is a free variable not occurring in the lower sequent

$$\exists\text{L}: \frac{\varphi(a) \Rightarrow \psi}{\exists x\varphi(x) \Rightarrow \psi} \qquad \exists\text{R}: \frac{\varphi \Rightarrow \psi(t)}{\varphi \Rightarrow \exists x\psi(x)}$$

a is a free variable not occurring in the lower sequent; t is any term

Note that in **LL**, there are no sequents like " $\Rightarrow \varphi$", "$\varphi \Rightarrow$ ", or " $\Rightarrow$ " for any φ.

5.2 Algebraic models of LL

We now define algebraic models $\mathcal{M} = \langle M, D\rangle$ of **LL**.

Definition 5.1 An algebraic model $\mathcal{M}$ of **LL** is a pair $\langle M, D\rangle$ satisfying the following conditions:

1. $M = \langle M, \wedge, \vee, \bigwedge, \bigvee, 0, 1\rangle$ is a complete lattice.
2. In order to define the truth value of a wff of the language L of **LL**, we need to define a valuation function

   $v : T_{\mathcal{M}} \longrightarrow D$ (D is a nonempty set and $T_{\mathcal{M}} := T_L \cup \{\bar{d} \mid d \in D\}$),

   where T_L is the set of all terms of the language L. $\bar{d} \in \{\bar{d} \mid d \in D\}$ is a new individual constant for $d \in D$. Thus, when we consider models of **LL**, the language L is to be extended by adding new individual constants $\{\bar{d} \mid d \in D\}$. v satisfies the two conditions:

   (1) for each $t \in T_L$, $v(t) \in D$
   (2) for each $d \in D$, $v(\bar{d}) := d$
3. For each wff φ of the extended language, its truth value $[\![\varphi]\!]^{\mathcal{M}}_v$ for $\mathcal{M}$ and v is defined as follows:

   (1) for each n-ary predicate symbol p, $p^{\mathcal{M}}$ is a function $p^{\mathcal{M}} : D^n \longrightarrow M$ and for

each atomic wff $p(t_1, \cdots, t_n)$,

$[\![p(t_1, \cdots, t_n)]\!]_v^{\mathcal{M}} := p^{\mathcal{M}}(v(t_1), \cdots, v(t_n))$

(2) $[\![\varphi \wedge \psi]\!]_v^{\mathcal{M}} := [\![\varphi]\!]_v^{\mathcal{M}} \wedge [\![\psi]\!]_v^{\mathcal{M}}$

(3) $[\![\varphi \vee \psi]\!]_v^{\mathcal{M}} := [\![\varphi]\!]_v^{\mathcal{M}} \vee [\![\psi]\!]_v^{\mathcal{M}}$

(4) $[\![\forall x\varphi(x)]\!]_v^{\mathcal{M}} := \bigwedge_{d\in D} [\![\varphi(\bar{d})]\!]_v^{\mathcal{M}}$

(5) $[\![\exists x\varphi(x)]\!]_v^{\mathcal{M}} := \bigvee_{d\in D} [\![\varphi(\bar{d})]\!]_v^{\mathcal{M}}$

For each model $\mathcal{M} = \langle M, D\rangle$ and valuation function v, the truth values of the propositional constants $\top$ and $\bot$ are set as follows:

$$[\![\top]\!]_v^{\mathcal{M}} := 1 \quad \text{and} \quad [\![\bot]\!]_v^{\mathcal{M}} := 0$$

4. We also define the truth value of a sequent $\varphi \Rightarrow \psi$ as follows:

$$[\![\varphi \Rightarrow \psi]\!]_v^{M} := \begin{cases} 1 & \text{if } [\![\varphi]\!]_v^{M} \leq [\![\psi]\!]_v^{M} \\ 0 & \text{otherwise} \end{cases}$$

5. The $\mathcal{M}$-validity and validity of a sequent $\varphi \Rightarrow \psi$ is defined as follows:

(1) $\varphi \Rightarrow \psi$ is $\mathcal{M}$-valid, $\mathcal{M} \models \varphi \Rightarrow \psi$, if $[\![\varphi \Rightarrow \psi]\!]_v^{\mathcal{M}} = 1$ for each v.

(2) $\varphi \Rightarrow \psi$ is valid, $\models \varphi \Rightarrow \psi$, if $\mathcal{M} \models \varphi \Rightarrow \psi$ holds for each $\mathcal{M}$.

Using the definition of a model for **LL**, we can show the soundness theorem for **LL**, the proof of which is routine:

Theorem 5.1 *Let φ and ψ be wffs of the language L of* **LL**. *Then we have*

$$\vdash \varphi \Rightarrow \psi \implies \models \varphi \Rightarrow \psi$$

We now define the Lindenbaum algebra of **LL**.

Definition 5.2 Let φ and ψ be wffs of the language L of **LL**. Then set

1. $\varphi \equiv \psi \overset{def}{\iff} (\vdash \varphi \Rightarrow \psi \text{ and } \vdash \psi \Rightarrow \varphi)$
2. $|\varphi| := \{\psi \in F_L \mid \varphi \equiv \psi\}$, where F_L is the set of all wffs of L.
3. $F_L/_\equiv := \{|\varphi| \mid \varphi \in F_L\}$
4. $|\varphi| \leq |\psi| \overset{def}{\iff} \vdash \varphi \Longrightarrow \psi$

We can easily show:

Proposition 5.2 *Of the above definition, the following hold* :

1. *The relation $\equiv$ on the wffs in F_L is an equivalence relation.*
2. *The relation $\leq$ on $F_L/_\equiv$ is well-defined.*
3. *$\langle F_L/_\equiv, \leq\rangle$ is a partially ordered set.*
4. *For each $\varphi, \psi \in F_L$,* $\inf\{|\varphi|, |\psi|\}$ *and* $\sup\{|\varphi|, |\psi|\}$ *exist and*

$$\inf\{|\varphi|, |\psi|\} = |\varphi \wedge \psi| \quad \textit{and} \quad \sup\{|\varphi|, |\psi|\} = |\varphi \vee \psi|$$

Then we set $|\varphi| \wedge |\psi| := \inf\{|\varphi|, |\psi|\}$ *and* $|\varphi| \vee |\psi| := \sup\{|\varphi|, |\psi|\}$.

5. $\langle F_L/_\equiv, \leq\rangle$ *is a lattice with a bottom* 0 *and a top* 1 *and*

$$|\bot| = 0 \quad \textit{and} \quad |\top| = 1$$

We also have

$$|\forall x\varphi(x)| = \bigwedge_{t\in T_L} |\varphi(t)| \quad \textit{and} \quad |\exists x\varphi(x)| = \bigvee_{t\in T_L} |\varphi(t)|$$

By the above proposition, we can call the algebra $F_L/_\equiv = \langle F_L/_\equiv, \wedge, \vee, 0, 1\rangle$ the Lindenbaum algebra of **LL**. Then we can prove the completeness theorem for **LL**:

Theorem 5.3 *Let* φ *and* ψ *be wffs of the language* L *of* **LL**. *Then we have*

$$\models \varphi \Rightarrow \psi \implies \vdash \varphi \Rightarrow \psi$$

Proof Suppose $\nvdash \varphi \Rightarrow \psi$. We need to show that the sequent $\varphi \Rightarrow \psi$ is not valid. From the Lindenbaum algebra of **LL**, $F_L/_\equiv = \langle F_L/_\equiv, \wedge, \vee, 0, 1\rangle$, we have $|\varphi| \not\leq |\psi|$. Let $(F_L/_\equiv)^{DM}$ be the Dedekind-MacNeille completion of $F_L/_\equiv$, i.e.,

$$(F_L/_\equiv)^{DM} := \{A \subseteq F_L/_\equiv \mid A^{ul} = A\},$$

where for each $B \subseteq F_L/_\equiv$,

$$B^u := \{x \in F_L/_\equiv \mid \forall b \in B\, (b \leq x)\} \quad \text{and} \quad B^l := \{x \in F_L/_\equiv \mid \forall b \in B\, (x \leq b)\}$$

It is well-known that $(F_L/_\equiv)^{DM} = \langle (F_L/_\equiv)^{DM}, \leq\rangle$, where the order $\leq$ is the set inclusion $\subseteq$, is a complete lattice and the mapping h

$$h : F_L/_\equiv \longrightarrow (F_L/_\equiv)^{DM} \text{ defined by } x \mapsto \downarrow x \quad (x \in F_L/_\equiv),$$

where $\downarrow x := \{a \in F_L/_\equiv \mid a \leq x\}$, is an order-embedding preserving all (finite or infinite) meets and joins. We now define an algebraic model $\mathcal{M} = \langle (F_L/_\equiv)^{DM}, D\rangle$ and a valuation function v as follows:

1. $D := T_L$ and $T_\mathcal{M} := T_L \cup D = T_L$
2. $v : T_\mathcal{M} \longrightarrow D$ is defined by $v(t) = t$ for each $t \in T_\mathcal{M} = D$
3. For each atomic wff $P(t_1, \cdots, t_n) \in F_L$, its truth value is defined by

$$[\![P(t_1, \cdots, t_n)]\!]^\mathcal{M}_v := h(|P(t_1, \cdots, t_n)|) = \downarrow |P(t_1, \cdots, t_n)| \in (F_L/_\equiv)^{DM}$$

Now we can show easily

Lemma For each wff $\varphi \in F_L$, $[\![\varphi]\!]^\mathcal{M}_v = h(|\varphi|)$

Since h is an order-embedding, we have, for each wff χ and ξ,

$$|\chi| \leq |\xi| \iff h(|\chi|) \leq h(|\xi|)$$

Thus we have $h(|\varphi|) \not\leq h(|\psi|)$. By the above lemma, we have $[\![\varphi]\!]_v^{\mathcal{M}} \not\leq [\![\psi]\!]_v^{\mathcal{M}}$, which gives us $\not\models \varphi \Rightarrow \psi$. $\square$

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References

(1) Aoyama, Hiroshi: "On a Weak System of Sequent Calculus", *Journal of Logical Philosophy* (this journal), no.3, 2003.

(2) Aoyama, Hiroshi: "LK, LJ, Dual Intuitionistic Logic, and Quantum Logic", *Notre Dame Journal of Formal Logic*, vol.45, no.4, 2004.

(3) Takeuti, Gaisi: "Proof Theory" (2nd ed.), North-Holland Publishing Co., Amsterdam, 1987.

(4) Urbas, Igor: "Dual-Intuitionistic Logic", *Notre Dame Journal of Formal Logic*, vol.37, no.3, 1996.

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