On a Weak System of Sequent Calculus

Hiroshi Aoyama (Tokaigakuen University)

Abstract In this paper, we study a very weak system of logic in the form of sequent calculus. The system, which we call "LB," is obtained from LK, i.e. Gentzen's sequent calculus of classical logic by putting some restriction on it. Our main concern in this paper is to clarify the syntactic relation between LK and LB and some basic semantic properties of LB.

1. Introduction

Formulating logical systems in the form of Gentzen's sequent calculus is attractive in that it is easy to see their logical structures and to compare them with each other from various viewpoints. In the present paper, we consider the logical system LB, which is obtained from LK by restricting sequents so that they contain at most one formula both in the antecedents and in the succedents. As is well-known, LJ, i.e. Gentzen's sequent calculus of intuitionistic logic, is obtained from LK by restricting sequents so that they contain at most one formula in the succedents. Similarly, we can obtain a dual system of LJ by restricting sequents of LK so that they contain at most one formula in the antecedents. It is often called the "dual-intuitionistic logic." We have already considered one such system named "DI."([1])

LB is a very weak system because it is a proper subsystem of both LJ and DI. Although it is weak, we need to clarify its syntactic and semantic properties. Before we start, we fix the language.

Definition 1.1. The language of the sequent calculi to be studied in this paper consists of the following symbols:

(1) Predicate constants with n argument-places ($n \geqq 0$): P_0^n, P_1^n, P_2^n, ...
(2) Free variables: a_0, a_1, a_2, ...
(3) Bound variables: x_0, x_1, x_2, ...
(4) Logical symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$
(5) Auxiliary symbols: (,) , ,(comma)

Well-formed formulas (wffs) are defined as usual. In a sequent $\Gamma \Rightarrow \Delta$, the antecedent Γ and the succedent Δ are both finite sequences of zero or more formulas, unless otherwise stated. Proofs (formal proofs) are also defined in the usual way. If a sequent $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus S , we write "S $\vdash \Gamma \Rightarrow \Delta$." We use $\alpha, \beta, \gamma, \delta, \pi$, and λ to express wffs and Γ, Δ, Λ, and Π to express sequences of wffs.

2. The system LB

Definition 2.1. LB has the following axioms and rules of inference:

(1) Axioms: $\alpha \Rightarrow \alpha$

(2) Inference Rules: (Γ and Δ consist of at most one wff.)

Structural Rules:

WL $$\frac{\Rightarrow \Delta}{\alpha \Rightarrow \Delta}$$

WR $$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$$

Cut $$\frac{\Gamma \Rightarrow \alpha \qquad \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Logical Rules: (a is a free variable.)

$\neg$L $$\frac{\Rightarrow \alpha}{\neg \alpha \Rightarrow}$$

$\neg$R $$\frac{\alpha \Rightarrow}{\Rightarrow \neg \alpha}$$

$\wedge$L $$\frac{\alpha \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta} \qquad \frac{\beta \Rightarrow \Delta}{\alpha \wedge \beta \Rightarrow \Delta}$$

$\wedge$R $$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

$\vee$L $$\frac{\alpha \Rightarrow \Delta \qquad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta}$$

$\vee$R $$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}$$

$\rightarrow$L $$\frac{\Rightarrow \alpha \qquad \beta \Rightarrow \Delta}{\alpha \rightarrow \beta \Rightarrow \Delta}$$

$\rightarrow$R $$\frac{\alpha \Rightarrow \beta}{\Rightarrow \alpha \rightarrow \beta}$$

$\forall$L $$\frac{\alpha(a) \Rightarrow \Delta}{\forall x\, \alpha(x) \Rightarrow \Delta}$$
(a is an arbitrary free variable.)

$\forall$R $$\frac{\Gamma \Rightarrow \alpha(a)}{\Gamma \Rightarrow \forall x\, \alpha(x)}$$
(a does not appear in the lower sequent.)

$\exists$L $$\frac{\alpha(a) \Rightarrow \Delta}{\exists x\, \alpha(x) \Rightarrow \Delta}$$
(a does not appear in the lower sequent.)

$\exists$R $$\frac{\Gamma \Rightarrow \alpha(a)}{\Gamma \Rightarrow \exists x\, \alpha(x)}$$
(a is an arbitrary free variable.)

We list some sequents which are provable in LB.

Proposition 2.2. The following sequents are provable in LB ("$\alpha \Leftrightarrow \beta$" indicates that $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. In (14), (16), (18), and (19), the variable x does not appear in β.):

(1) $\Rightarrow \alpha \rightarrow \alpha$

(2) $\neg(\alpha \rightarrow \alpha) \Rightarrow$

(3) $\alpha \Leftrightarrow \alpha \wedge (\alpha \vee \beta)$

(4) $\alpha \Leftrightarrow \alpha \vee (\alpha \wedge \beta)$

(5) $\alpha \wedge \beta \Rightarrow \alpha \wedge (\neg \alpha \vee \beta)$

(6) $\alpha \vee (\neg \alpha \wedge \beta) \Rightarrow \alpha \vee \beta$

(7) $(\alpha \wedge \beta) \vee (\alpha \wedge \neg \beta) \Rightarrow \alpha$

(8) $\alpha \Rightarrow (\alpha \vee \beta) \wedge (\alpha \vee \neg \beta)$

(9) $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$

(10) $\alpha \vee (\beta \wedge \gamma) \Rightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

(11) $(\alpha \wedge \beta) \vee (\gamma \wedge \delta) \Rightarrow (\alpha \vee \gamma) \wedge (\alpha \vee \delta) \wedge (\beta \vee \gamma) \wedge (\beta \vee \delta)$

(12) $(\alpha \wedge \gamma) \vee (\alpha \wedge \delta) \vee (\beta \wedge \gamma) \vee (\beta \vee \delta) \Rightarrow (\alpha \vee \beta) \wedge (\gamma \vee \delta)$

(13) $\forall x\, \alpha(x) \Rightarrow \exists x\, \alpha(x)$

(14) $\forall x(\alpha(x) \wedge \beta) \Leftrightarrow \forall x\, \alpha(x) \wedge \beta$

(15) $\forall x(\alpha(x) \wedge \beta(x)) \Leftrightarrow \forall x\, \alpha(x) \wedge \forall x\, \beta(x)$

(16) $\exists x(\alpha(x) \vee \beta) \Leftrightarrow \exists x\, \alpha(x) \vee \beta$

(17) $\exists x(\alpha(x) \vee \beta(x)) \Leftrightarrow \exists x\, \alpha(x) \vee \exists x\, \beta(x)$

(18) $\forall x\alpha(x)\vee\beta \Rightarrow \forall x(\alpha(x)\vee\beta)$ (19) $\exists x(\alpha(x)\wedge\beta) \Rightarrow \exists x\alpha(x)\wedge\beta$

We also have some derived inference rules.

PROPOSITION 2.3. The following derived rules of inference hold in LB:

(1) $\dfrac{\Rightarrow\ \alpha}{\Rightarrow\ \neg\neg\alpha}$ (2) $\dfrac{\alpha\ \Rightarrow}{\neg\neg\alpha\ \Rightarrow}$

(3) $\dfrac{\Rightarrow\ \alpha \qquad \Rightarrow\ \alpha\to\beta}{\Rightarrow\ \beta}$

As a sequent calculus, LB satisfies the cut elimination theorem.

THEOREM 2.4. The rule Cut is eliminable from LB.

PROOF. The proof is routine. ■

COROLLARY 2.5. LB has the subformula property and it is consistent.

Since LB is a subsystem of both intuitionistic logic and dual-intuitionistic logic, there are many important sequents which are not in general provable in it. We list some such:

$\alpha\Rightarrow\neg\neg\alpha$, $\neg\neg\alpha\Rightarrow\alpha$, $\Rightarrow\alpha\vee\neg\alpha$, $\alpha\wedge\neg\alpha\Rightarrow$, $\alpha\Rightarrow(\beta\to\alpha)$,
$\alpha\wedge(\alpha\to\beta)\Rightarrow\beta$, $\neg\alpha\to\neg\beta\Rightarrow\beta\to\alpha$, $\alpha\wedge(\beta\vee\gamma)\Rightarrow(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$

There are, of course, sequents of the form " $\Rightarrow\alpha\vee\neg\alpha$" or "$\alpha\wedge\neg\alpha\Rightarrow$," which are provable in LB such as " $\Rightarrow(\alpha\to\alpha)\vee\neg(\alpha\to\alpha)$" and "$(\alpha\to\alpha)\wedge\neg(\alpha\to\alpha)\Rightarrow$."

3. An Equivalent System of LB

We now consider another formulation of LB, which we call "LB′." LB′ is more like LK in that there is no special restriction on the number of formulas in sequents. LB′ is related to LB as LJ′, Maehara's intuitionistic sequent calculus, is to Gentzen's LJ. The main goal of this section is to show the equivalence of LB and LB′.

DEFINITION 3.1. **LB′** has the following axioms and rules of inference:

(1) Axioms: $\alpha\ \Rightarrow\ \alpha$

(2) Inference Rules: (There is no restriction on Γ, Δ, Π, and Λ except that they are finite.)

Structural Rules:

WL $\dfrac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}$ WR $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$

CL $\dfrac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}$ CR $\dfrac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$

EL $\dfrac{\Gamma, \alpha, \beta, \Pi \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Pi \Rightarrow \Delta}$ ER $\dfrac{\Gamma \Rightarrow \Delta, \alpha, \beta, \Lambda}{\Gamma \Rightarrow \Delta, \beta, \alpha, \Lambda}$

Cut 1 $\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda}$ Cut 2 $\dfrac{\Gamma \Rightarrow \alpha \qquad \alpha, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}$

Logical Rules: (a is a free variable.)

$\neg$L $\dfrac{\Rightarrow \alpha}{\neg \alpha \Rightarrow}$ $\neg$R $\dfrac{\alpha \Rightarrow}{\Rightarrow \neg \alpha}$

$\wedge$L $\dfrac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$ $\dfrac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$ $\wedge$R $\dfrac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$

$\vee$L $\dfrac{\alpha \Rightarrow \Delta \qquad \beta \Rightarrow \Delta}{\alpha \vee \beta \Rightarrow \Delta}$ $\vee$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$ $\dfrac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$

$\rightarrow$L $\dfrac{\Rightarrow \alpha \qquad \beta \Rightarrow \Delta}{\alpha \rightarrow \beta \Rightarrow \Delta}$ $\rightarrow$R $\dfrac{\alpha \Rightarrow \beta}{\Rightarrow \alpha \rightarrow \beta}$

$\forall$L $\dfrac{\alpha(a), \Gamma \Rightarrow \Delta}{\forall x\, \alpha(x), \Gamma \Rightarrow \Delta}$ $\forall$R $\dfrac{\Gamma \Rightarrow \alpha(a)}{\Gamma \Rightarrow \forall x\, \alpha(x)}$

(a is an arbitrary free variable.) (a does not appear in the lower sequent.)

$\exists$L $\dfrac{\alpha(a) \Rightarrow \Delta}{\exists x\, \alpha(x) \Rightarrow \Delta}$ $\exists$R $\dfrac{\Gamma \Rightarrow \Delta, \alpha(a)}{\Gamma \Rightarrow \Delta, \exists x\, \alpha(x)}$

(a does not appear in the lower sequent.) (a is an arbitrary free variable.)

In what follows, "$\cdot\,\Gamma$" indicates the conjunction of all the wffs in Γ and if Γ is empty, so is $\cdot\,\Gamma$. Similarly, "$\cdot\,\Delta$" indicates the disjunction of all the wffs in Δ and if Δ is empty, so is $\cdot\,\Delta$. Then we have

Proposition 3.2. $LB' \vdash \Gamma \Rightarrow \Delta$ iff $LB' \vdash \cdot\,\Gamma \Rightarrow \cdot\,\Delta$.

Proof. We can prove this by using the inference rules $\wedge$L, $\wedge$R, $\vee$L, and $\vee$R of LB′.

■

Now we can show the equivalence of LB and LB′.

Proposition 3.3. $LB \vdash \cdot\,\Gamma \Rightarrow \cdot\,\Delta$ iff $LB' \vdash \cdot\,\Gamma \Rightarrow \cdot\,\Delta$.

PROOF. If the sequent $\cdot\,\Gamma \Rightarrow \cdot\,\Delta$ is provable in LB, then it is also provable in LB′ because the former calculus is a subsystem of the latter.

For the other direction, it is enough to show that all the inference rules of LB′ holds in LB. We consider the two rules Cut 1 and $\forall$L. Let us first set $\gamma = \cdot\,\Gamma$, $\delta = \cdot\,\Delta$, and $\lambda = \cdot\,\Lambda$.

Cut 1 of LB′ can be rewritten by the previous proposition as: $\dfrac{\gamma \Rightarrow \delta \vee \alpha \qquad \alpha \Rightarrow \lambda}{\gamma \Rightarrow \delta \vee \lambda}$

This holds in LB as follows:

$$\dfrac{\gamma \Rightarrow \delta \vee \alpha \qquad \dfrac{\dfrac{\delta \Rightarrow \delta}{\delta \Rightarrow \delta \vee \lambda} \qquad \dfrac{\alpha \Rightarrow \lambda}{\alpha \Rightarrow \delta \vee \lambda}}{\delta \vee \alpha \Rightarrow \delta \vee \lambda}}{\gamma \Rightarrow \delta \vee \lambda}$$

$\forall$L of LB′ can be rewritten as: $\dfrac{\alpha(a) \wedge \gamma \Rightarrow \delta}{\forall x\, \alpha(x) \wedge \gamma \Rightarrow \delta}$, where a is a free variable.

Using (14) of Proposition 2.2, it is easy to see that this rule holds in LB:

$$\dfrac{\forall x\, \alpha(x) \wedge \gamma \Rightarrow \forall x(\alpha(x) \wedge \gamma) \qquad \dfrac{\alpha(a) \wedge \gamma \Rightarrow \delta}{\forall x(\alpha(x) \wedge \gamma) \Rightarrow \delta}}{\forall x\, \alpha(x) \wedge \gamma \Rightarrow \delta}$$

■

4. LB′ and LK

In this section, we show the equivalence of LK and an extension of LB′, which we call “LBc.”

DEFINITION 4.1. LBc is obtained from LB′ by adding as extra axioms the following:

(1) $\Rightarrow \alpha \vee \neg \alpha$
(2) $\alpha \wedge \neg \alpha \Rightarrow$
(3) $\alpha \rightarrow \beta \Rightarrow \neg \alpha \vee \beta$
(4) $\neg \alpha \vee \beta \Rightarrow \alpha \rightarrow \beta$
(5) $\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
(6) $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \Rightarrow \alpha \vee (\beta \wedge \gamma)$
(7) $\forall x(\alpha \vee \beta(x)) \Rightarrow \alpha \vee \forall x\, \beta(x)$
(8) $\alpha \wedge \exists x\, \beta(x) \Rightarrow \exists x(\alpha \wedge \beta(x))$

In (7) and (8), the variable x does not appear in α.

We then show that LBc is equivalent to LK.

THEOREM 4.2. LK $\vdash \cdot\,\Gamma \Rightarrow \cdot\,\Delta$ iff LBc $\vdash \cdot\,\Gamma \Rightarrow \cdot\,\Delta$.

PROOF. The direction from right to left is clear because LB′ is a subsystem of

LK and the eight extra axioms in the previous definition are provable in LK.

For the direction from left to right, it is enough to show that all the inference rules of LK hold in LBc. We set $\gamma =\cdot\ \Gamma$, $\pi =\cdot\ \Pi$, $\delta =\cdot\ \Delta$, and $\lambda =\cdot\ \Lambda$. The structural rules WL, WR, CL, CR, EL, and ER of LK are the same as those of LBc.

(1) $\neg$L $\dfrac{\gamma \Rightarrow \delta \vee \alpha}{\neg\alpha \wedge \gamma \Rightarrow \delta}$: This inference rule of LK holds in LBc as is shown below:

(In what follows, we sometimes omit writing easy applications of inference rules.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\alpha \Rightarrow \neg\alpha}{\neg\alpha\wedge\gamma \Rightarrow \neg\alpha} \quad \dfrac{\gamma \Rightarrow \delta\vee\alpha}{\neg\alpha\wedge\gamma \Rightarrow \delta\vee\alpha}}{\neg\alpha\wedge\gamma \Rightarrow \neg\alpha\wedge(\delta\vee\alpha)} \quad \neg\alpha\wedge(\delta\vee\alpha) \Rightarrow (\neg\alpha\wedge\delta)\vee(\neg\alpha\wedge\alpha)}{\neg\alpha\wedge\gamma \Rightarrow (\neg\alpha\wedge\delta)\vee(\neg\alpha\wedge\alpha)} \quad \dfrac{\neg\alpha\wedge\delta \Rightarrow \neg\alpha\wedge\delta \quad \dfrac{\neg\alpha\wedge\alpha \Rightarrow}{\neg\alpha\wedge\alpha \Rightarrow \neg\alpha\wedge\delta}}{(\neg\alpha\wedge\delta)\vee(\neg\alpha\wedge\alpha) \Rightarrow \neg\alpha\wedge\delta}}{\neg\alpha\wedge\gamma \Rightarrow \neg\alpha\wedge\delta} \quad \neg\alpha\wedge\delta \Rightarrow \delta}{\neg\alpha\wedge\gamma \Rightarrow \delta}$$

(2) $\neg$R $\dfrac{\alpha \wedge \gamma \Rightarrow \delta}{\gamma \Rightarrow \delta \vee \neg\alpha}$: This holds in LBc as follows:

$$\dfrac{\gamma \Rightarrow \gamma\vee\neg\alpha \quad \dfrac{\dfrac{\dfrac{\Rightarrow \alpha\vee\neg\alpha}{\gamma\vee\neg\alpha \Rightarrow \alpha\vee\neg\alpha} \quad \gamma\vee\neg\alpha \Rightarrow \gamma\vee\neg\alpha}{\gamma\vee\neg\alpha \Rightarrow (\alpha\vee\neg\alpha)\wedge(\gamma\vee\neg\alpha)} \quad \dfrac{(\alpha\vee\neg\alpha)\wedge(\gamma\vee\neg\alpha) \Rightarrow (\alpha\wedge\gamma)\vee\neg\alpha \quad \dfrac{\dfrac{\alpha\wedge\gamma \Rightarrow \delta}{\alpha\wedge\gamma \Rightarrow \delta\vee\neg\alpha} \quad \dfrac{\neg\alpha \Rightarrow \neg\alpha}{\neg\alpha \Rightarrow \delta\vee\neg\alpha}}{(\alpha\wedge\gamma)\vee\neg\alpha \Rightarrow \delta\vee\neg\alpha}}{(\alpha\vee\neg\alpha)\wedge(\gamma\vee\neg\alpha) \Rightarrow \delta\vee\neg\alpha}}{\gamma\vee\neg\alpha \Rightarrow \delta\vee\neg\alpha}}{\gamma \Rightarrow \delta\vee\neg\alpha}$$

(3) The case of $\wedge$L of LK is easy and the distributive laws of LBc are enough for the cases of $\wedge$R and $\vee$L. The rule of $\vee$R of LK is basically the same as that of LBc.

(4) $\to$L $\dfrac{\gamma \Rightarrow \delta \vee \alpha \qquad \beta \wedge \pi \Rightarrow \lambda}{(\alpha \to \beta) \wedge \gamma \wedge \pi \Rightarrow \delta \vee \lambda}$:

Using $\neg$L and $\vee$L of LK above, it is easy to see that this rule of LK holds in LBc:

$$\dfrac{\dfrac{\dfrac{\alpha\to\beta \Rightarrow \neg\alpha\vee\beta}{(\alpha\to\beta)\wedge\gamma\wedge\pi \Rightarrow \neg\alpha\vee\beta} \quad \dfrac{\gamma\wedge\pi \Rightarrow \gamma\wedge\pi}{(\alpha\to\beta)\wedge\gamma\wedge\pi \Rightarrow \gamma\wedge\pi}}{(\alpha\to\beta)\wedge\gamma\wedge\pi \Rightarrow (\neg\alpha\vee\beta)\wedge\gamma\wedge\pi} \quad \dfrac{\dfrac{\dfrac{\dfrac{\gamma\Rightarrow\delta\vee\alpha}{\neg\alpha\wedge\gamma\Rightarrow\delta}}{\neg\alpha\wedge\gamma\Rightarrow\delta\vee\lambda}}{\neg\alpha\wedge\gamma\wedge\pi \Rightarrow \delta\vee\lambda} \quad \dfrac{\dfrac{\beta\wedge\pi\Rightarrow\lambda}{\beta\wedge\pi\Rightarrow\delta\vee\lambda}}{\beta\wedge\gamma\wedge\pi \Rightarrow \delta\vee\lambda}}{(\neg\alpha\vee\beta)\wedge\gamma\wedge\pi \Rightarrow \delta\vee\lambda}}{(\alpha\to\beta)\wedge\gamma\wedge\pi \Rightarrow \delta\vee\lambda}$$

(5) $\to$R $\dfrac{\alpha \wedge \gamma \Rightarrow \delta \vee \beta}{\gamma \Rightarrow \delta \vee (\alpha \to \beta)}$: Using $\neg$R of LK above, we have this:

$$\dfrac{\dfrac{\dfrac{\alpha\wedge\gamma\Rightarrow\delta\vee\beta}{\gamma\Rightarrow\delta\vee\beta\vee\neg\alpha}\qquad \delta\vee\beta\vee\neg\alpha\Rightarrow\delta\vee\neg\alpha\vee\beta}{\gamma\Rightarrow\delta\vee\neg\alpha\vee\beta}\qquad \dfrac{\dfrac{\delta\Rightarrow\delta}{\delta\Rightarrow\delta\vee(\alpha\to\beta)}\qquad\dfrac{\neg\alpha\vee\beta\Rightarrow\alpha\to\beta}{\neg\alpha\vee\beta\Rightarrow\delta\vee(\alpha\to\beta)}}{\delta\vee\neg\alpha\vee\beta\Rightarrow\delta\vee(\alpha\to\beta)}}{\gamma\Rightarrow\delta\vee(\alpha\to\beta)}$$

(6) Cut $\dfrac{\gamma\Rightarrow\delta\vee\alpha\qquad\alpha\wedge\pi\Rightarrow\lambda}{\gamma\wedge\pi\Rightarrow\delta\vee\lambda}$:

Using $\neg$L and $\neg$R of LK above, this rule of LK holds in LBc:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\pi\wedge\pi\Rightarrow}{\pi\Rightarrow\neg\neg\pi}}{\gamma\wedge\pi\Rightarrow\neg\neg\pi}\qquad\gamma\wedge\pi\Rightarrow\gamma}{\gamma\wedge\pi\Rightarrow\neg\neg\pi\wedge\gamma}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma\Rightarrow\delta\vee\alpha\quad\delta\vee\alpha\Rightarrow\alpha\vee\delta}{\gamma\Rightarrow\alpha\vee\delta}}{\neg\delta\wedge\gamma\Rightarrow\alpha}\qquad\dfrac{\dfrac{\pi\wedge\alpha\Rightarrow\alpha\wedge\pi\quad\alpha\wedge\pi\Rightarrow\lambda}{\pi\wedge\alpha\Rightarrow\lambda}}{\alpha\Rightarrow\lambda\vee\neg\pi}}{\neg\delta\wedge\gamma\Rightarrow\lambda\vee\neg\pi}}{\gamma\Rightarrow\lambda\vee\neg\pi\vee\neg\neg\delta}\qquad\dfrac{\lambda\vee\neg\pi\Rightarrow\lambda\vee\neg\pi\vee\delta\qquad\dfrac{\dfrac{\Rightarrow\delta\vee\neg\delta}{\neg\neg\delta\Rightarrow\delta}}{\neg\neg\delta\Rightarrow\lambda\vee\neg\pi\vee\delta}}{\lambda\vee\neg\pi\vee\neg\neg\delta\Rightarrow\lambda\vee\neg\pi\vee\delta}}{\gamma\Rightarrow\lambda\vee\neg\pi\vee\delta}\qquad\lambda\vee\neg\pi\vee\delta\Rightarrow\delta\vee\lambda\vee\neg\pi}{\gamma\Rightarrow\delta\vee\lambda\vee\neg\pi}}{\neg\neg\pi\wedge\gamma\Rightarrow\delta\vee\lambda}}{\gamma\wedge\pi\Rightarrow\delta\vee\lambda}$$

(7) For the quantification rules of LK, we only show the rule of $\forall$R and the rest are similar.

$\forall$R $\dfrac{\gamma\Rightarrow\delta\vee\alpha(a)}{\gamma\Rightarrow\delta\vee\forall x\,\alpha(x)}$, where the variable a does not appear in the lower sequent.

This is simple:

$$\dfrac{\dfrac{\gamma\Rightarrow\delta\vee\alpha(a)}{\gamma\Rightarrow\forall x(\delta\vee\alpha(x))}\qquad\forall x(\delta\vee\alpha(x))\Rightarrow\delta\vee\forall x\,\alpha(x)}{\gamma\Rightarrow\delta\vee\forall x\,\alpha(x)}$$

■

Since LK $\vdash\Gamma\Rightarrow\Delta$ is equivalent to LK $\vdash\cdot\ \Gamma\Rightarrow\cdot\ \Delta$, we have

Corollary 4.3. LK $\vdash\Gamma\Rightarrow\Delta$ iff LBc $\vdash\ \cdot\ \Gamma\Rightarrow\cdot\ \Delta$.

5. A Brief Consideration on the Semantics of LB

In this section, we briefly consider an algebraic semantics for LB. It is easy to see that the axioms and inference rules of LB satisfies the properties of a complete lattice and the following conditions, where $\neg$ is a unary operation and $\to$ is a binary operation on the complete lattice and a, b, and c denote its arbitrary elements:

(1) $\neg a = 1$ if $a = 0$,

$= 0 \quad \text{if} \quad a = 1.$

(2) (i) $a \rightarrow b = 1$ if $a \leq b$.
(ii) $a \rightarrow b \leq c$ if $a = 1$ and $b \leq c$.

The condition (1) corresponds to the inference rules $\neg$R and $\neg$L of LB and (2) corresponds to $\rightarrow$R and $\rightarrow$L, respectively. The second part of (2) can be also expressed as

$$1 \rightarrow b \leq c \quad \text{if} \quad b \leq c.$$

We can now present the definition of an algebraic structure for LB.

Definition 5.1. $L = \langle L, \leq, \wedge, \vee, \neg, \rightarrow, \bigwedge, \bigvee, 0, 1 \rangle$ is a structure for LB if the following three conditions are satisfied:

(1) $\langle L, \leq, \wedge, \vee, \bigwedge, \bigvee, 0, 1 \rangle$ is a complete lattice.
(2) $\neg$ is a unary operation on L satisfying: for each $a \in L$,

$$\neg a = 1 \quad \text{if} \quad a = 0,$$
$$= 0 \quad \text{if} \quad a = 1.$$

(3) $\rightarrow$ is a binary operation on L satisfying: for each $a, b, c \in L$,

(i) $a \rightarrow b = 1$ if $a \leq b$.
(ii) $a \rightarrow b \leq c$ if $a = 1$ and $b \leq c$.

We call such a structure a "cLBa (complete LB algebra)." For the operations $\neg$ and $\rightarrow$ of a cLBa, we can easily obtain the following: for each $a, b \in L$,

(1) $1 \rightarrow a \leq a$ (2) $a \rightarrow 1 = 1$ (3) $0 \rightarrow a = 1$
(4) $a \rightarrow b \leq \neg a \vee b$ if $a = 1$ (5) $a \rightarrow b = \neg a \vee b$ if $b = 1$

We now define model structures for LB.

Definition 5.2. $M = \langle L, D, f \rangle$ is a model structure for LB if the following conditions are satisfied:

1. L is a cLBa.
2. D is a nonempty set.
3. f is a function on the set of predicate symbols of LB; that is, for each n-ary predicate symbol P^n, $f(P^n)\colon D^n \rightarrow L$.
4. Formulas of LB are interpreted in M recursively as follows:
   Let FV be the set of free variables of LB. v expresses a function from FV to D. For $a \in$ FV, v[d/a] expresses a function like v except that it assigns d of D to a. We write the interpretation of a formula α with respect to M and v as "$(M,v) \cdot \alpha$." For each atomic formula $P^n(a_1,\ldots,a_n)$, we set

   (1) $(M,v) \cdot P^n(a_1,\ldots,a_n) = f(P^n)(v(a_1),\ldots,v(a_n))$.

For compound formulas, we set

(2) $(M,\mathrm{v}) \cdot (\alpha \wedge \beta) = ((M,\mathrm{v}) \cdot \alpha \wedge (M,\mathrm{v}) \cdot \beta)$.
(3) $(M,\mathrm{v}) \cdot (\alpha \vee \beta) = ((M,\mathrm{v}) \cdot \alpha \vee (M,\mathrm{v}) \cdot \beta)$.
(4) $(M,\mathrm{v}) \cdot \neg\alpha = 1$ if $(M,\mathrm{v}) \cdot \alpha = 0$,
$= 0$ if $(M,\mathrm{v}) \cdot \alpha = 1$.
(5) $(M,\mathrm{v}) \cdot (\alpha \rightarrow \beta) = 1$ if $(M,\mathrm{v}) \cdot \alpha \leq (M,\mathrm{v}) \cdot \beta$. Also,
$(M,\mathrm{v}) \cdot (\alpha \rightarrow \beta) \leq (M,\mathrm{v}) \cdot \gamma$ if $(M,\mathrm{v}) \cdot \alpha = 1$ and $(M,\mathrm{v}) \cdot \beta \leq (M,\mathrm{v}) \cdot \gamma$.
(6) $(M,\mathrm{v}) \cdot \forall x \alpha(x) = \bigwedge\{(M,\mathrm{v}[\mathrm{d}/\mathrm{a}]) \cdot \alpha(\mathrm{a}) \mid \mathrm{d} \in D\}$.
(7) $(M,\mathrm{v}) \cdot \exists x \alpha(x) = \bigvee\{(M,\mathrm{v}[\mathrm{d}/\mathrm{a}]) \cdot \alpha(\mathrm{a}) \mid \mathrm{d} \in D\}$.

5. A sequent "$\alpha \Rightarrow \beta$" is valid if , for every $M = \langle L, D, \mathrm{f} \rangle$ and every v, $(M,\mathrm{v}) \cdot \alpha \leq (M,\mathrm{v}) \cdot \beta$.
Similarly, "$\Rightarrow \alpha$" is valid if , for every $M = \langle L, D, \mathrm{f} \rangle$ and every v, $(M,\mathrm{v}) \cdot \alpha = 1$.
Also, "$\alpha \Rightarrow$ " is valid if , for every $M = \langle L, D, \mathrm{f} \rangle$ and every v, $(M,\mathrm{v}) \cdot \alpha = 0$. We write "$\cdot\ \Gamma \Rightarrow \Delta$" to express the validity of the sequent.

Now we can easily show the soundness theorem for LB with respect to the above algebraic model structures.

THEOREM 5.3.(Soundness Theorem) If LB $\vdash \Gamma \Rightarrow \Delta$, then $\cdot\ \Gamma \Rightarrow \Delta$.

PROOF. Proof by induction on the length of the proof of $\Gamma \Rightarrow \Delta$.

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6. Concluding Remarks

This is an introductory paper on the system LB and there remain many problems, syntactical or semantical, about it. Proof-theoretically, we need to investigate how to form LJ and DI from LB. It is also interesting to study intermediate logics between LB and LJ and also those between LB and DI. Model-theoretically, we need to construct Kripke models for LB. It might also be interesting to study how much of mathematics can be developed in LB. In any case, since the logical operators $\neg$ and $\rightarrow$ play a crucial role in LB, we need to study them more carefully.

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